

Modified Gravity Theories Based on the Non-Canonical Volume-Form Formalism

D. Benisty,^{1,2,*} E. I. Guendelman,^{1,2,3,†} A. Kaganovich,^{1,‡} E. Nissimov,^{4,§} and S. Pacheva^{4,¶}

¹Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

²Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany

³Bahamas Advanced Study Institute and Conferences, 4A Ocean Heights, Hill View Circle, Stella Maris, Long Island, The Bahamas

⁴Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria

(Received May 27, 2019)

This is a shortened version of an invited talk at the XIII International Workshop “Lie Theory and its Applications in Physics”, June 17-23, Varna, Bulgaria. We present a concise description of the basic features of gravity-matter models based on the formalism of non-canonical spacetime volume-forms in its two versions: the *method of non-Riemannian volume-forms* (metric-independent covariant volume elements) and the *dynamical spacetime formalism*. Among the principal outcomes we briefly discuss: (i) quintessential universe evolution with a gravity-“inflaton”-assisted suppression in the “early” universe and, respectively, dynamical generation in the “late” universe of Higgs spontaneous electroweak gauge symmetry breaking; (ii) unified description of dark energy and dark matter as manifestations of a single material entity – a second scalar field “darkon”; (iii) unification of dark energy and dark matter with diffusive interaction among them; (iv) explicit derivation of a stable “emergent universe” solution, *i.e.*, a creation without Big Bang; (v) mechanism for suppression of 5-th force without fine-tuning.

1. Non-Riemannian Volume-Form Formalism - Extended (modified) gravity theories as alternatives/generalizations of the standard Einstein General Relativity (for detailed accounts, see Refs. [1]-[4]) are being widely studied in the last decade or so due to pressing motivation from cosmology (problems of dark energy and dark matter), quantum field theory in curved spacetime (renormalization in higher loops) and string theory (low-energy effective field theories).

A broad class of actively developed modified/extended gravitational theories is based on employing alternative non-Riemannian spacetime volume-forms (metric-independent generally covariant volume elements) in the pertinent Lagrangian actions instead of the canonical Riemannian one given by the square-root of the determinant of the Riemannian metric (originally proposed in [5, 6], for a concise geometric formulation, see [7, 8]). A characteristic feature of these extended gravitational theories is that when starting in the first-order (Palatini) formalism the non-Riemannian volume-forms are almost *pure-gauge* degrees of freedom, *i.e.* they *do not* introduce any additional propagating gravitational degrees of freedom except for few discrete degrees of freedom appearing as arbitrary integration constants (for a canonical Hamiltonian treatment, see Appendices A in Refs.[8, 9]).

Let us recall that volume-forms in integrals over differentiable manifolds (not necessarily Riemannian one, so *no* metric is needed) are given by nonsingular maximal rank differential forms ω :

$$\int_{\mathcal{M}} \omega(\dots) = \int_{\mathcal{M}} dx^D \Omega(\dots) ,$$

$$\omega = \frac{1}{D!} \omega_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D} , \quad (1)$$

$$\omega_{\mu_1 \dots \mu_D} = -\varepsilon_{\mu_1 \dots \mu_D} \Omega ,$$

(our conventions for the alternating symbols $\varepsilon^{\mu_1, \dots, \mu_D}$ and $\varepsilon_{\mu_1, \dots, \mu_D}$ are: $\varepsilon^{01 \dots D-1} = 1$ and $\varepsilon_{01 \dots D-1} = -1$). The volume element Ω transforms as scalar density under general coordinate reparametrizations.

In standard generally-covariant theories (with action $S = \int d^D x \sqrt{-g} \mathcal{L}$) the Riemannian spacetime volume-form is defined through the “D-bein” (frame-bundle) canonical one-forms $e^A = e^A_{\mu} dx^{\mu}$ ($A = 0, \dots, D-1$):

$$\omega = e^0 \wedge \dots \wedge e^{D-1} = \det \|e^A_{\mu}\| dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D}$$

$$\rightarrow \Omega = \det \|e^A_{\mu}\| d^D x = \sqrt{-\det \|g_{\mu\nu}\|} d^D x . \quad (2)$$

Instead of $\sqrt{-g}$ we can employ another alternative *non-Riemannian* volume element as in (1) given by a non-singular *exact* D -form $\omega = dB$ where:

$$B = \frac{1}{(D-1)!} B_{\mu_1 \dots \mu_{D-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{D-1}}$$

$$\rightarrow \Omega \equiv \Phi(B) = \frac{1}{(D-1)!} \varepsilon^{\mu_1 \dots \mu_D} \partial_{\mu_1} B_{\mu_2 \dots \mu_D} . \quad (3)$$

In other words, the non-Riemannian volume element is defined in terms of the dual field-strength of an auxiliary rank $D-1$ tensor gauge field $B_{\mu_1 \dots \mu_{D-1}}$.

To illustrate the main interesting properties of the new class of extended gravity-matter models based on the non-Riemannian volume-form formalism we will consider gravity in the Palatini formalism coupled in a non-standard way via non-Riemannian volume elements to [9–11]: (i) scalar “inflaton” field φ ; (ii) a second scalar “darkon” field u ; (iii) the bosonic fields of the standard electroweak particle model – $\sigma \equiv (\sigma_a)$ being a complex $SU(2) \times U(1)$ iso-doublet Higgs-like scalar, and the $SU(2) \times U(1)$ gauge fields \vec{A}_{μ}, B_{μ} .

The ‘‘inflaton’’ φ apart from driving the cosmological evolution triggers suppression, respectively, generation of the electroweak (Higgs) spontaneous symmetry breaking in the ‘‘early’’, respectively, in the ‘‘late’’ universe. The ‘‘darkon’’ u is responsible for the unified description of dark energy and dark matter in the ‘‘late’’ universe.

The corresponding action reads (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$\begin{aligned} S = & \int d^4x \Phi_1(A) \left[R + L^{(1)}(\varphi, \sigma) \right] \\ & + \int d^4x \Phi_2(B) \left[L^{(2)}(\varphi, \vec{\mathcal{A}}, \vec{\mathcal{B}}) + \frac{\Phi_4(H)}{\sqrt{-g}} \right] \\ & - \int d^4x (\sqrt{-g} + \Phi_3(C)) \frac{1}{2} g^{\mu\nu} \partial_\mu u \partial_\nu u . \end{aligned} \quad (4)$$

Here the following notations are used:

(i) $\Phi_1(A), \Phi_2(B), \Phi_3(C)$ are three independent non-Riemannian volume elements as in (3) for $D = 4$; $\Phi_4(H)$ is again of the form (3) for $D = 4$ and it is needed for consistency of (4).

(ii) The scalar curvature R in Palatini formalism is $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$, where the Ricci tensor is a function of the affine connection $\Gamma_{\mu\nu}^\lambda$ *a priori* independent of $g_{\mu\nu}$.

(iii) The matter field Lagrangians are:

$$\begin{aligned} L^{(1)}(\varphi, \sigma) \equiv & -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - f_1 e^{-\alpha\varphi} \\ & - g^{\mu\nu} (\nabla_\mu \sigma)_a^* \nabla_\nu \sigma_a - \frac{\lambda}{4} ((\sigma_a)^* \sigma_a - \mu^2)^2 , \end{aligned} \quad (5)$$

$$\begin{aligned} L^{(2)}(\varphi, \vec{\mathcal{A}}, \vec{\mathcal{B}}) = & -\frac{b}{2} e^{-\alpha\varphi} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + f_2 e^{-2\alpha\varphi} \\ & - \frac{1}{4g^2} F^2(\vec{\mathcal{A}}) - \frac{1}{4g'^2} F^2(\vec{\mathcal{B}}) , \end{aligned} \quad (6)$$

where α, f_1, f_2 are dimensionful positive parameters, whereas b is a dimensionless one (b is needed to obtain a stable ‘‘emergent’’ universe solution, see below (25)). $F^2(\vec{\mathcal{A}})$ and $F^2(\vec{\mathcal{B}})$ in (6) are the squares of the field-strengths of the electroweak gauge fields, and the last term in (5) is of the same form as the standard Higgs potential.

Let us note that the form of the ‘‘inflaton’’ part of the action (4) is fixed by the requirement of invariance under global Weyl-scale transformations:

$$\begin{aligned} g_{\mu\nu} & \rightarrow \lambda g_{\mu\nu} , \quad \Gamma_{\nu\lambda}^\mu \rightarrow \Gamma_{\nu\lambda}^\mu , \quad \varphi \rightarrow \varphi + \frac{1}{\alpha} \ln \lambda , \\ A_{\mu\nu\kappa} & \rightarrow \lambda A_{\mu\nu\kappa} , \quad B_{\mu\nu\kappa} \rightarrow \lambda^2 B_{\mu\nu\kappa} , \quad H_{\mu\nu\kappa} \rightarrow H_{\mu\nu\kappa} . \end{aligned} \quad (7)$$

Scale invariance played an important role in the original papers on the non-canonical volume-form formalism where also fermions were included [6] (see also Section 3 below).

The equations of motion of the initial action (4) w.r.t. auxiliary tensor gauge fields $A_{\mu\nu\lambda}, B_{\mu\nu\lambda}, C_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda}$

yield the following algebraic constraints:

$$\begin{aligned} R + L^{(1)} = M_1 = \text{const} , \quad L^{(2)} + \frac{\Phi_4(H)}{\sqrt{-g}} = -M_2 = \text{const} , \\ -\frac{1}{2} g^{\mu\nu} \partial_\mu u \partial_\nu u = M_0 = \text{const} , \quad \frac{\Phi_2(B)}{\sqrt{-g}} \equiv \chi_2 = \text{const} , \end{aligned} \quad (8)$$

where M_0, M_1, M_2 are arbitrary dimensionful and χ_2 an arbitrary dimensionless *integration constants*.

The equations of motion of (4) w.r.t. affine connection $\Gamma_{\nu\lambda}^\mu$ yield a solution for $\Gamma_{\nu\lambda}^\mu$ as a Levi-Civita connection $\Gamma_{\nu\lambda}^\mu = \Gamma_{\nu\lambda}^\mu(\bar{g}) = \frac{1}{2} \bar{g}^{\mu\kappa} (\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda})$ w.r.t. to the a *Weyl-rescaled metric* $\bar{g}_{\mu\nu} = \chi_1 g_{\mu\nu}$, $\chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}}$.

The passage to the ‘‘Einstein-frame’’ (EF) is accomplished by a Weyl-conformal transformation to $\bar{g}_{\mu\nu}$ upon using relations (8), so that the EF action with a canonical Hilbert-Einstein gravity part w.r.t. $\bar{g}_{\mu\nu}$ and with the canonical Riemannian volume element $\sqrt{\det || - \bar{g}_{\mu\nu} ||}$ reads:

$$S_{\text{EF}} = \int d^4x \sqrt{-\bar{g}} \left[R(\bar{g}) + L_{\text{EF}} \right] , \quad (9)$$

and where the EF matter Lagrangian turns out to be of a quadratic ‘‘k-essence’’ type [12]-[15] w.r.t. both the ‘‘inflaton’’ φ and ‘‘darkon’’ u fields:

$$\begin{aligned} L_{\text{EF}} = & \bar{X} - \bar{Y} \left[f_1 e^{-\alpha\varphi} + \frac{\lambda}{4} ((\sigma_a)^* \sigma_a - \mu^2)^2 + M_1 \right. \\ & \left. - \chi_2 b e^{-\alpha\varphi} \bar{X} \right] + \bar{Y}^2 \left[\chi_2 (f_2 e^{-2\alpha\varphi} + M_2) + M_0 \right] \\ & + L[\sigma, \vec{\mathcal{A}}, \vec{\mathcal{B}}] , \end{aligned} \quad (10)$$

with $L[\sigma, \vec{\mathcal{A}}, \vec{\mathcal{B}}] \equiv -\bar{g}^{\mu\nu} (\nabla_\mu \sigma_a)^* \nabla_\nu \sigma_a - \frac{\chi_2}{4g'^2} \bar{F}^2(\vec{\mathcal{A}}) - \frac{\chi_2}{4g'^2} \bar{F}^2(\vec{\mathcal{B}})$. In (10) all quantities defined in terms of the EF metric $\bar{g}_{\mu\nu}$ are indicated by an upper bar, and the following short-hand notations are used: $\bar{X} \equiv -\frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$, $\bar{Y} \equiv -\frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu u \partial_\nu u$.

From (10) we deduce the following full effective scalar potential:

$$U_{\text{eff}}(\varphi, \sigma) = \frac{\left(f_1 e^{-\alpha\varphi} + \frac{\lambda}{4} ((\sigma_a)^* \sigma_a - \mu^2)^2 + M_1 \right)^2}{4 \left[\chi_2 (f_2 e^{-2\alpha\varphi} + M_2) + M_0 \right]} \quad (11)$$

As discussed in Refs.[10, 11] $U_{\text{eff}}(\varphi, \sigma)$ (11) has few remarkable properties. First, $\mathcal{U}_{\text{total}}(\varphi, \sigma)$ possesses two infinitely large flat regions as function of φ when σ is fixed:

- (a) (-) flat ‘‘inflaton’’ region for large negative values of φ corresponding to the evolution of the ‘‘early’’ universe;
- (b) (+) flat ‘‘inflaton’’ region for large positive values of φ with σ fixed corresponding to the evolution of the ‘‘late’’ universe’.

In the (-) flat ‘‘inflaton’’ region, *i.e.*, in the ‘‘early’’ universe the effective scalar field potential (11) reduces

to (an approximately) constant value

$$U_{\text{eff}}(\varphi, \sigma) \simeq U_{(-)} = \frac{f_1^2}{4\chi_2 f_2} \quad (12)$$

Thus, there is no σ -field potential and, therefore, *no electroweak spontaneous breakdown in the “early” universe.*

On the other hand, in the (+) flat “inflaton” region, *i.e.*, in the “late” universe the effective scalar field potential becomes:

$$U_{\text{eff}}(\varphi, \sigma) \simeq U_{(+)}(\sigma) = \frac{\left(\frac{\lambda}{4}((\sigma_a)^* \sigma_a - \mu^2)^2 + M_1\right)^2}{4(\chi_2 M_2 + M_0)}, \quad (13)$$

which obviously yields *nontrivial vacuum for the Higgs-like field* $|\sigma_{\text{vac}}| = \mu$. Therefore, in the “late” universe we have the standard spontaneous breakdown of electroweak $SU(2) \times U(1)$ gauge symmetry. Moreover, at the Higgs vacuum we obtain from (13) a dynamically generated cosmological constant $\Lambda_{(+)}$ of the “late” Universe:

$$U_{(+)}(\mu) \equiv 2\Lambda_{(+)} = \frac{M_1^2}{4(\chi_2 M_2 + M_0)}. \quad (14)$$

If we identify the integration constants with the fundamental scales in Nature as $M_{0,1} \sim M_{EW}^4$ and $M_2 \sim M_{Pl}^4$, where where M_{Pl} is the Planck mass scale and $M_{EW} \sim 10^{-16} M_{Pl}$ is the electroweak mass scale, then $\Lambda_{(+)} \sim M_{EW}^8/M_{Pl}^4 \sim 10^{-120} M_{Pl}^4$, which is the right order of magnitude for the present epoch’s vacuum energy density as already realized in [16].

On the other hand, if we take the order of magnitude of the coupling constants in the effective potential (11) $f_1 \sim f_2 \sim (10^{-2} M_{Pl})^4$, then the order of magnitude of the vacuum energy density of the “early” universe (12) becomes:

$$U_{(-)} \sim f_1^2/f_2 \sim 10^{-8} M_{Pl}^4, \quad (15)$$

which conforms to the Planck Collaboration data [17, 18] implying the energy scale of inflation of order $10^{-2} M_{Pl}$.

Now, performing FLRW reduction of the EF action (9) we obtain in the “late” universe, *i.e.*, for large positive “inflaton” φ values the following results for the density, pressure, the Friedmann scale factor (the solution for $a(t)$ below first appeared in [19]) and the “inflaton” velocity:

$$\rho = \frac{M_1^2}{4(\chi_2 M_2 + M_0)} + \frac{\pi_u}{a^3} \left[\frac{M_1}{\chi_2 M_2 + M_0} \right]^{\frac{1}{2}} + O\left(\frac{\pi_u^2}{a^6}\right) \quad (16)$$

$$p = -\frac{M_1^2}{4(\chi_2 M_2 + M_0)} + O\left(\frac{\pi_u^2}{a^6}\right) \quad (17)$$

$$a(t) \simeq \left(\frac{C_0}{2\Lambda_{(+)}}\right)^{1/3} \sinh^{2/3}\left(\sqrt{\frac{3}{4}\Lambda_{(+)}} t\right), \quad (18)$$

$$\dot{\varphi} \simeq \text{const} \sinh^{-2}\left(\sqrt{\frac{3}{4}\Lambda_{(+)}} t\right), \quad (19)$$

where π_u is the conserved “darkon” canonical momentum, $\Lambda_{(+)}$ is as in (14) and $C_0 \equiv \pi_u \sqrt{M_1(\chi_2 M_2 + M_0)^{-1}}$.

Relations (16)-(17) straightforwardly show that in the “late” universe we have explicit unification of dark energy (given by the dynamically generated cosmological constant (14) – first constant terms on the r.h.sides in (16) and (17), and dark matter given as a “dust” fluid contribution – second term $O(a^{-3})$ on the r.h.s. of (16).

A further interesting property under consideration is the existence of a stable “emergent” universe solution – a creation without Big Bang (cf. Refs.[21, 22]). It is characterized by the condition on the Hubble parameter H :

$$H = 0 \quad \rightarrow \quad a(t) = a_0 = \text{const}, \quad \rho + 3p = 0, \quad \frac{K}{a_0^2} = \frac{1}{6}\rho (= \text{const}), \quad (20)$$

and the “inflaton” is on the (–) flat region (large negative values of φ). Then relations (20) together with the “inflaton” and “darkon” equations of motion imply that also “inflaton” velocity $\dot{\varphi} = \text{const}$ and the constant density and pressure read:

$$\rho \simeq -\frac{3\chi_2 b^2}{16f_2} \dot{\varphi}^4 - \frac{1}{2} \dot{\varphi}^2 \left(1 + \frac{bf_1}{2f_2}\right) + \frac{f_1^2}{4\chi_2 f_2}, \quad (21)$$

$$p \simeq -\frac{\chi_2 b^2}{16f_2} \dot{\varphi}^4 - \frac{1}{2} \dot{\varphi}^2 \left(1 + \frac{bf_1}{2f_2}\right) - \frac{f_1^2}{4\chi_2 f_2}. \quad (22)$$

The truncated Friedmann Eqs.(20) yield exact solutions for the constant “inflaton” velocity $\dot{\varphi}_0$ and Friedmann factor a_0 :

$$\dot{\varphi}_0^2 = \frac{8f_2}{3\chi_2 b^2} \left[1 + \frac{bf_1}{2f_2} - \sqrt{\left(1 + \frac{bf_1}{2f_2}\right)^2 - \frac{3b^2 f_1^2}{16f_2^2}}\right], \quad (23)$$

and $a_0^2 = 6K/\rho_0$ where:

$$\rho_0 = \frac{f_1^2}{2\chi_2 f_2} - \frac{1}{2} \dot{\varphi}_0^2 \left(1 + \frac{bf_1}{2f_2}\right). \quad (24)$$

Studying perturbation $a \rightarrow a + \delta a(t)$ of the “emergent” universe condition (20) we obtain a harmonic oscillator equation for $\delta a(t)$ (here $\dot{\varphi}_0^2$ as in (23)):

$$\delta \ddot{a} + \omega^2 \delta a = 0,$$

$$\omega^2 \equiv \frac{\rho_0}{6} \left[3 - \frac{\frac{1}{2}(1 + bf_1/2f_2) - \dot{\varphi}_0^2 \chi_2 b^2/8f_2}{\dot{\varphi}_0^2 3\chi_2 b^2/8f_2 - \frac{1}{2}(1 + bf_1/2f_2)} - 1\right] > 0 \quad (25)$$

for $-8(1 - \frac{1}{2}\sqrt{3})\frac{f_2}{f_1} < b < -\frac{f_2}{f_1}$.

The non-Riemannian volume-form formalism was also successfully applied to propose a qualitatively new mechanism for a *dynamical spontaneous breaking of supersymmetry* in supergravity by constructing modified formulation of standard minimal $N = 1$ supergravity

as well as of anti-de Sitter supergravity in terms of a non-Riemannian volume elements [7, 20]. This naturally triggers the appearance of a *dynamically generated cosmological constant* as an arbitrary integration constant which signifies *dynamical spontaneous supersymmetry breakdown*. The same formalism applied to anti-de Sitter supergravity allows us to appropriately choose the above mentioned arbitrary integration constant so as to obtain simultaneously a *very small effective observable cosmological constant* as well as a *large physical gravitino mass* as required by modern cosmological scenarios for slowly expanding universe of the present epoch [23–25].

2. Dynamical Spacetime formulation - Let us now observe that the non-Riemannian volume element $\Omega = \Phi(B)$ (3) on a Riemannian manifold can be rewritten using Hodge duality (here $D = 4$) in terms of a vector field $\chi^\mu = \frac{1}{3!} \frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\kappa\lambda} B_{\nu\kappa\lambda}$ so that Ω becomes $\Omega(\chi) = \partial_\mu(\sqrt{-g}\chi^\mu)$, i.e. it is a non-canonical volume element different from $\sqrt{-g}$, but involving the metric. It can be represented alternatively through a Lagrangian multiplier action term yielding covariant conservation of a specific energy-momentum tensor of the form $\mathcal{T}^{\mu\nu} = g^{\mu\nu}\mathcal{L}$:

$$\mathcal{S}_{(\chi)} = \int d^4x \sqrt{-g} \chi_{\mu;\nu} \mathcal{T}^{\mu\nu} = \int d^4x \partial_\mu(\sqrt{-g}\chi^\mu)(-\mathcal{L}), \quad (26)$$

where $\chi_{\mu;\nu} = \partial_\nu \chi_\mu - \Gamma_{\mu\nu}^\lambda \chi_\lambda$.

The vector field χ_μ is called “*dynamical space time vector*”, because of the energy density of \mathcal{T}^{00} is a canonically conjugated momentum w.r.t. χ_0 , which is what we expected from a dynamical time.

In what follows we will briefly consider a new class of gravity-matter theories based on the ordinary Riemannian volume element $\sqrt{-g}$ but involving action terms of the form (26) where now $\mathcal{T}^{\mu\nu}$ is of more general form than $\mathcal{T}^{\mu\nu} = g^{\mu\nu}\mathcal{L}$. This new formalism is called “*dynamical spacetime formalism*” [26, 27] due to the above remark on χ_0 .

Ref.[28] describes a unification between dark energy and dark matter by introducing a quintessential scalar field in addition to the dynamical time action. The total Lagrangian reads:

$$\mathcal{L} = \frac{1}{2}R + \chi_{\mu;\nu}\mathcal{T}^{\mu\nu} - \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi), \quad (27)$$

with energy-momentum tensor $\mathcal{T}^{\mu\nu} = -\frac{1}{2}\phi^{,\mu}\phi^{,\nu}$. From the variation of the Lagrangian term $\chi_{\mu;\nu}\mathcal{T}^{\mu\nu}$ with respect to the vector field χ_μ , the covariant conservation of the energy-momentum tensor $\nabla_\mu\mathcal{T}^{\mu\nu} = 0$ is implemented. The latter within the FLRW framework forces the kinetic term of the scalar field to behave as a dark matter component:

$$\nabla_\mu\mathcal{T}^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\phi}^2 = \frac{2\Omega_{m0}}{a^3}. \quad (28)$$

where Ω_{m0} is an integration constant. The variation with respect to the scalar field ϕ yields a current:

$$-V'(\phi) = \nabla_\mu j^\mu, \quad j^\mu = \frac{1}{2}\phi_{,\nu}(\chi^{\mu;\nu} + \chi^{\nu;\mu}) + \phi^{,\mu} \quad (29)$$

For constant potential $V(\phi) = \Omega_\Lambda = \text{const}$ the current is covariantly conserved.

In the FLRW setting, where the dynamical time ansatz introduces only a time component $\chi_\mu = (\chi_0, 0, 0, 0)$, the variation (29) gives:

$$\dot{\chi}_0 - 1 = \xi a^{-3/2}, \quad (30)$$

where ξ is an integration constant. Accordingly, the FLRW energy density and pressure read:

$$\rho = \left(\dot{\chi}_0 - \frac{1}{2}\right)\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2(\dot{\chi}_0 - 1) - V. \quad (31)$$

Plugging the relations (28,30) into the density and the pressure terms (31) yields the following simple form of the latter:

$$\rho = \Omega_\Lambda + \frac{\xi\Omega_{m0}}{a^{9/2}} + \frac{\Omega_{m0}}{a^3}, \quad p = -\Omega_\Lambda + \frac{\xi\Omega_{m0}}{2a^{9/2}}. \quad (32)$$

In (32) there are 3 components for the “dark fluid”: dark energy with $\omega_\Lambda = -1$, dark matter with $\omega_m = 0$ and an additional equation of state $\omega_\xi = 1/2$. For non-vanishing and negative ξ the additional part introduces a minimal scale parameter, which avoids singularities. If the dynamical time is equivalent to the cosmic time $\chi_0 = t$, we obtain $\xi = 0$ from Eq.(30), whereupon the density and the pressure terms (32) coincide with those from the Λ CDM model precisely. The additional part (for $\xi \neq 0$) fits more to the late time accelerated expansion data, as observed in Ref. [29].

Ref. [30] shows that with higher dimensions, the solution derived from the Lagrangian (27) describes inflation, where the total volume oscillates and the original scale parameter exponentially *grows*.

The dynamical spacetime Lagrangian can be generalized to yield a *diffusive energy-momentum tensor*. Ref. [31] shows that the diffusion equation has the form:

$$\nabla_\mu\mathcal{T}^{\mu\nu} = 3\sigma j^\nu, \quad j^\mu_{;\mu} = 0, \quad (33)$$

where σ is the diffusion coefficient and j^μ is a current source. The covariant conservation of the current source indicates the conservation of the number of the particles. By introducing the vector field χ_μ in a different part of the Lagrangian:

$$\mathcal{L}_{(\chi,A)} = \chi_{\mu;\nu}\mathcal{T}^{\mu\nu} + \frac{\sigma}{2}(\chi_\mu + \partial_\mu A)^2, \quad (34)$$

the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ gets a *diffusive source*. From a variation with respect to the dynamical space time vector field χ_μ we obtain:

$$\nabla_\nu\mathcal{T}^{\mu\nu} = \sigma(\chi^\mu + \partial^\mu A) = f^\mu, \quad (35)$$

a current source $f^\mu = \sigma(\chi^\mu + \partial^\mu A)$ for the energy-momentum tensor. From the variation with respect to the new scalar A , a covariant conservation of the current emerges $f_{;\mu}^\mu = 0$. *The latter relations correspond to the diffusion equation (33).*

Refs.[33–36] study the cosmological solution using the energy-momentum tensor $\mathcal{T}^{\mu\nu} = -\frac{1}{2}g^{\mu\nu}\phi_{;\lambda}\phi_{;\lambda}$. The total Lagrangian reads:

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - V(\phi) + \chi_{\mu;\nu}\mathcal{T}^{\mu\nu} + \frac{\sigma}{2}(\chi_\mu + \partial_\mu A)^2. \quad (36)$$

The FLRW solution unifies the dark energy and the dark matter originating from one scalar field with possible diffusion interaction. Ref.[32] investigates more general energy-momentum tensor combinations and shows that asymptotically all of the combinations yield Λ CDM model as a stable fixed point.

Scale Invariance, Fifth Force and Fermionic Matter - The originally proposed theory with two volume elements (integration measure densities) [5, 6], where at least one of them was a non-canonical one and short-termed “two-measure theory” (TMT), has a number of remarkable properties if fermions are included in a self-consistent way [6]. In this case, the constraint that arises in the TMT models in the Palatini formalism can be represented as an equation for $\chi \equiv \Phi/\sqrt{-g}$, in which the left side has an order of the vacuum energy density, and the right side (in the case of non-relativistic fermions) is proportional to the fermion density. Moreover, it turns out that even cold fermions have a (non-canonical) pressure P_f^{noncan} and the corresponding contribution to the energy-momentum tensor has the structure of a cosmological constant term which is proportional to the fermion density. The remarkable fact is that the right hand side of the constraint coincide with P_f^{noncan} . This allows us to construct a cosmological model[37] of the late universe in which dark energy is generated by a gas of non-relativistic neutrinos without the need to introduce into the model a specially designed scalar field.

In models with a scalar field, the requirement of scale invariance of the initial action[5] plays a very constructive role. It allows to construct a model[38] where without fine tuning we have realized: absence of initial singularity of the curvature; k-essence; inflation with graceful exit to zero cosmological constant.

Of particular interest are scale invariant models in which both fermions and a dilaton scalar field ϕ are present. Then it turns out that the Yukawa coupling of fermions to ϕ is proportional to P_f^{noncan} . As a result, it follows from the constraint, that in all cases when fermions are in states which constitute a regular barionic matter, the Yukawa coupling of fermions to dilaton has an order of ratio of the vacuum energy density to the

fermion energy density[39]. Thus, the theory provides a solution of the 5-th force problem without any fine tuning or a special design of the model. Besides, in the described states, the regular Einstein’s equations are reproduced. In the opposite case, when fermions are very deluted, e.g. in the model of the late Universe filled with a cold neutrino gas, the neutrino dark energy appears in such a way that the dilaton ϕ dynamics is closely correlated with that of the neutrino gas[39].

A scale invariant model containing a dilaton ϕ and dust (as a model of matter)[40] possesses similar features. The dilaton to matter coupling “constant” f appears to be dependent of the matter density. In normal conditions, i.e. when the matter energy density is many orders of magnitude larger than the dilaton contribution to the dark energy density, f becomes less than the ratio of the “mass of the vacuum” in the volume occupied by the matter to the Planck mass. The model yields this kind of “Archimedes law” without any especial (intended for this) choice of the underlying action and without fine tuning of the parameters. The model not only explains why all attempts to discover a scalar force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter) and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

Acknowledgements. We gratefully acknowledge support of our collaboration through the academic exchange agreement between the Ben-Gurion University in Beer-Sheva, Israel, and the Bulgarian Academy of Sciences. E.N. and E.G. have received partial support from European COST actions MP-1405, CA-16104, CA-18108 and from CA-15117, CA-16104, CA-18108 respectively. E.N. and S.P. are also thankful to Bulgarian National Science Fund for support via research grant DN-18/1.

* Electronic address: benidav@post.bgu.ac.il

† Electronic address: guendel@bgu.ac.il

‡ Electronic address: alexanderkgnvch2@gmail.com

§ Electronic address: nissimov@inrne.bas.bg

¶ Electronic address: svetlana@inrne.bas.bg

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